ANALYZING THE FORMATION OF GROUPS IN A NETWORK ADAPTING THE MODULARITY CONCEPT

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Abstract

It is known that nodes in a community are more likely to connect to other members of the same community than to nodes in other communities and that new nodes tend to connect to nodes with higher degree increasing their centrality and following preferential attachment.

It is possible the manipulation of the peripheral nodes (nodes with less centrality) annexation logic belonging to a specific group – not necessarily a community – avoiding the preferential attachment logic and favoring other nodes in the same group at the expense of nodes with greater centrality. This is called "strategic behavior". The group of peripheral nodes adopting the strategic behavior does not necessarily form a distinct community from the other nodes; rather, they are inserted in the community as peripheral nodes. Therefore, there is room to look for a modification of the methods for communities' identification so that it is possible to identify the emergence of strategic groups.

The concept of networks modularity partially applies to the context described. In this paper, it is proposed a measure, adapted from the modularity concept, to test if this concept can identify the existence of strategic groups, assuming that the group of nodes adopting this strategic behavior is known. To build on this setting, a network was generated using a Stochastic Block Model (SBM) algorithm with two classes, in a way that in one of them (the strategic group) the nodes were linked with greater probability and there was no distinction in the probability of connections between classes, simulating this strategic behavior. Also, a method to identify this specific behavior based on the number of links inside the groups and between them during time is introduced. A computational simulation method was implemented to measure the adapted modularity and to model the method proposed here, to confirm the existence of the strategic group.

Key words: social networks, modularity, community detection, centrality, Stochastic Block Model,simulation.

Introduction

The problem of cutting a graph into "useful" sub graphs is classical in graph theory (DUGUÉ and PEREZ, 2015). In several fields from social networks to bioinformatics, the task of detecting these sub graphs efficiently constitutes a very relevant research field (DUCH and ARENAS, 2005). In most cases, the analyzed graphs representing data are directed. (NICOSIA et. al., 2009).

Due to the diversity of areas that represent different kinds of structures in which the graphs may be useful there are different reasons and motivations for dividing graphs into smaller components. Those kind of structures are very interesting, both for theoretical and practical reasons. One of them is because they naturally arise as from simple interactions among people and do not require sophisticated mechanisms to arise and be preserved. Besides that, they have some useful properties, such as high internal connectivity, low path length among nodes and high robustness, which are of great importance in real applications (NICOSIA et. al., 2009). Therefore, a lot of mathematical literature on methods to solve this problem has been produced, showing that these "clustering algorithms" can be able to optimize a graph structure in order to guarantee these certain desired properties.

Although, studies performed in the last few years showed that algorithms used for graph clustering are not very useful for explaining partitioning patterns observed in social networks, such as the arising of "communities", "groups" or "clubs" (NICOSIA et. al., 2009).

There is not a precise definition of "community" (NICOSIA et. al., 2009). One of the most widely accepted and used is that given by Newman and Girvan (2004):

> "a community is a sub graph containing nodes which are more densely linked to each other than to the rest of the graph or, equivalently, a graph has a community structure if the number of links into any sub graph is higher than the number of links between those sub graphs".

Indeed, the given definition of communities is a reasonable one if we consider the fact that real–life communities are groups of strongly connected nodes, as happens with people in a football club, authors in a co–authorship paper or colleagues studying in the same school (NEWMAN and GIRVAN, 2003; NEWMAN, 2006). It is important to highlight that usually nodes in a community know each other, so the probability for two nodes in a community to have a neighbor in common is higher than for other nodes in the graph outside the community (NEWMAN and JUYONG, 2003; NICOSIA et. al., 2009).

Considering that, a social network is an example of a network where we can find communities or groups. A social network has a growth dynamics governed by the principle of preferential attachment, i.e. more central nodes have a greater power of attraction for new connections.

It is known that nodes in a community are more likely to connect to other members of the same community than to nodes in other communities and that new nodes tend to connect to nodes with higher degree, increasing their centrality and following the preferential attachment (BARABÁSI, 2015). Therefore, it is observed a center-periphery structure: a group formed by few agents (center) who receive a disproportionate amount of connections coexisting with another set formed by the other agents that receive few connections (periphery). For nodes with less centrality (peripheral nodes) the centrality gain, which is measured in degree of entrance, is a great challenge (SANDE, 2016).

With that in mind, it is plausible to suppose that the peripheral nodes may perceive that they can influence other less central nodes, in the sense that links are created exclusively to increase their centrality and also their importance. Thus, a theoretical possibility raised is the emergence of an influence of the peripheral nodes on each other, in order to induce them to follow a decision-making logic of allocation of links that favors the increase of reciprocity[1] with some other peripheral nodes. Therefore, while observing the presence of relevant and necessary links to construct the natural relations between, the expectation would be that the links resulting from this strategy would be able to give them a positive effect in their importance to the network. This strategy is called Strategic Behavior (SANDE, 2016).

Basically, the Strategic Behavior is the creation of new links between peripheral nodes increasing the representativeness of the connections between two specific nodes in relation to the others and deviating from the logic of the preferential attachment. Applying to a social network, some peripheral nodes start linking to each other to increase their importance measured by centrality, but these nodes continue to be seen by the rest of the network as ordinary nodes (SANDE, 2016). Hence, the nodes adopting this strategic behavior can be seen as a "strategic group" and the other nodes of the network as a "non-strategic group".

[1] Reciprocity of links between two nodes P1 and P2 is the existence of links of P1 that are directed to P2, which are matched by links of P2 that are directed to P1 (SANDE, 2016).

The strategic group does not necessarily form a distinct community from the other nodes; rather, their nodes are inserted in the graph, and their in-degree selection behavior has no difference from the other network nodes, while their out-degree selection behavior follows the logic of a community. Therefore, there is room to look for a modification of the methods for communities' identification, so that it is possible to identify the emergence of strategic groups.

Based on that, this study aims to generate a random graph with two groups (called here strategic and non-strategic) through a Stochastic Block Model (SBM) algorithm and calculate values of "submodularity" to verify if the strategic behavior can be identified by the modularity concept. In a next stage, it is suggested a method to identify this kind of behavior by observing photographs of a simulated social network at different time intervals to verify the increase and/or drop of links within and between the groups, analyzing the communities in pairs, confirming that the strategic behavior occurs.

1 Modularity

A classic method for detecting communities is to find a partition of the node set that maximizes an optimization function called modularity (NEWMAN, 2003). This function values the existence of an edge between two vertices of an undirected network by comparing it with the probability of having such an edge in a random graph following the same degree distribution of the original network (DUGUÉ and PEREZ, 2015). That is, for a given division of the network's vertices into some modules, modularity reflects the concentration of edges within modules compared with random distribution of edges between all nodes regardless of modules. Formally, the modularity Q of a partition C of an undirected graph is defined as follows:

$$Q = \frac{1}{2m} \sum_{vw} \left[A_{vw} - \frac{k_v k_w}{2m} \right] \delta(c_v, c_w) \tag{1}$$

where m is the number of edges of the graph, A_{vw} represents the weight of the edge between v and w (set to 0 if such an edge does not exist), k_v is the degree of vertex v (i.e. the number of neighbors of v), c_v is the community to which vertex v belongs and the δ-function $\delta(x, y)$ is defined as 1 if $x = y$, and 0 otherwise (DUGUÉ and PEREZ, 2015).

Taking into account that a social network has a directional character, i.e. a link clearly identifies the node where the link comes from and to which one it goes to, it is necessary to adapt the notion of modularity to this kind of graph.

Leicht and Newman (2008) adapted the concept of modularity for directed graphs motivated by the fact that if two vertices v and w have small in-degree/large out-degree and small out-degree/large in-degree, respectively, then having an edge directed from w to v should be considered more surprising than having one from v to w. Considering that, the definition for directed modularity of a partition of a directed network can be formulated by:

$$Q = \frac{1}{m} \sum_{vw} \left[A_{vw} - \frac{k_v^{in} k_w^{out}}{m} \right] \delta(c_v c_w) \tag{2}$$

where A_{vw} now represents the existence of an edge between v and w and k_v^{in}/ k_w^{out}stands for the in-degree/out-degree of v (DUGUÉ and PEREZ, 2015).

Considering the partition of the graph in R groups, we have:

$$\delta(c_v c_w) = \sum_r S_{vr} S_{wr} \tag{3}$$

where S_{vr} = 1 if v belongs to group r ($0<r<R$) and zero otherwise. Then we have for directed graphs:

$$Q = \frac{1}{m}\sum_{vw}\sum_{r}\left[A_{vw} - \frac{k_v k_w}{m}\right] S_{vr}S_{wr} \quad (4)$$

Rewriting:

$$Q = \frac{1}{m}\sum_{vw} B_{vw} s_v s_w \quad (5)$$

where:

$$B_{vw} = A_{vw} - \frac{k_v^{in} k_w^{out}}{m} \quad (6)$$

Simplifying:

$$Q = \frac{1}{m} Tr(S^T B S) \quad (7)$$

where *Tr* represents the trace of the matrix.
For directed networks divided into just two communities, S_{vr} assumes only two values (+1/-1), and the matrix *S* becomes a vector *s*, leading to:

$$\mathrm{Q} = \frac{1}{\mathrm{m}} \mathrm{S}^{\mathrm{T}}\mathrm{BS} \quad (8)$$

By the definition of modularity it is clear that it is clear that it only considers that there is a tendency that the links inside the communities increase and that ones between communities decrease. Indeed, the existent algorithms to maximize the modularity function work regarding to the fact that there are a lot of links inside the communities than between them. Nevertheless, when a network has a strategic group it is needed to take into account that is expected a augmentation in the number of links inside the strategic group during time, but the links between groups should remain practically the same. Therefore, the simple concept of modularity does not identify this behavior, so it is proposed in this work an adaptation in the modularity concept to try to identify it, called here “submodularity”, which will be explained in the next section.

Another important concept used in this study is the Stochastic Block Model. The SBM is a generative model for random graphs. It tends to produce graphs containing communities, where links may be more common within communities than between them. The model takes the following parameters:

- the number *n* of vertices;
- a partition of the vertex set {1,...,*n*} into disjoint *K* subsets $\{c_1,..., c_R\}$, called communities;
- a symmetric K*xK* matrix *P* of edge probabilities.

The edge set is then sampled at random as follows: any two vertices are connected by an edge with probability P_{ij}.

In the proposed study, as explained next, we have a directed network divided in two groups (strategic and non-strategic), so the calculations are based in the value of *Q* expressed by equation 8, *K*=2 and *P* is a 2x2 matrix.

2 Proposed Study

In this study, an adapted measure of modularity is proposed to identify the existence of strategic groups, assuming that the group of nodes adopting this strategic behavior is known. This adapted modularity was suggested, since that the simple modularity concept is only able to identify explicit communities and the strategic group is not necessarily a community, as told before.

To build on this setting, a network was generated using a SBM algorithm with two classes: strategic group and non-strategic group. The links between nodes belonging to the strategic group were set with greater probability than the links between nodes belonging to the non-strategic group and between nodes belonging to distinct classes, and the probability for links among non-strategic nodes was the same as for the links between distinct classes' nodes, simulating this strategic behavior.

The first step is to generate a random graph with two groups (strategic, group 1, and non-strategic, group 2) through the SBM and than calculate the "submodularity" varying the configuration of the partition of the nodes of the graphs. Two values of "submodularity" are proposed here: Q_1 and Q_2, where Q_1 refers to links inside the strategic group – which has a higher probability; and Q_2 refers to the links between the two groups (directed from group 2 to group 1) – which has a lower probability, as group 2 does not see the nodes from group 1 as a group, but with no distinction from the rest of the network:

$$Q_1 = \frac{1}{m} s_1^T B s_1 \qquad (9)$$

$$Q_2 = \frac{1}{m} s_1^T B s_2 \qquad (10)$$

Where s_1 is defined as:

- s_{1v} = 1, if node v is starategic (belongs to group 1);
- s_{1v} = 0, if node v is non-strategic;

and s_2 is defined as:

- s_{2v} = 1, if node v is non-strategic (belongs to group 2);
- s_{2v} = 0, if node v is strategic.

To prove that a community exists, it is needed that the value of the modularity to be positive. Taking this into account, the following features are needed to identify the strategic group:

- $Q_1 > 0$ (strategic)
- $Q_2 \sim 0$ (non-strategic)

Simulations with 400 and 2000 nodes were made where the inputs to generate the Adjacency Matrix (A) of the graph through the SBM were:

- Matrix of probabilities (P) – with the probabilities of nodes to connect inside a group and between groups (2x2); and
- Partition vector (c) – indicates if a node belongs or not to the strategic group (1xn).

Based on A, the values of Q_1 and Q_2 were calculated.

But, observing the definitions of Q_1 and Q_2, we can easily see that they are symmetric:

$$Q_1 + Q_2 = \frac{1}{m}(\, S_1^T B S_1 + S_1^T B S_2) =$$
$$= \frac{1}{m}(\, S_1^T B(S_1 + S_2) = S_1^T \left(A - \frac{\overrightarrow{k_{in}}\,\overrightarrow{k_{out}}}{m} \right) \vec{u} =$$
$$= S_1^T \left(A\vec{u} - \overrightarrow{k_{in}} \frac{\overrightarrow{k_{out}}}{m} \vec{u} \right) = S_1^T \left(\overrightarrow{k_{in}} - \overrightarrow{k_{in}} \right) = 0,$$

Where $(S_1+S_2) = \vec{u}$ = unit vector nx1. Then:

$$Q_1 + Q_2 = 0 \rightarrow Q_1 = -Q_2$$

So, one of the goals of this study is to show in a practical way that even the "submodularity" concept proposed here is not able to identify the strategic behavior.

With that in mind, it is also suggested here that to identify this specific behavior, it is necessary to observe the network at different time stamps to verify the variation of the number of links inside and between groups, considering that the total number of links in the graph remains constant.

To measure the variation of links between and inside the two groups during time, was used the following reasons, where c_{ij} represents the number of links from group j to i (i, j = 1,2):

$$\alpha = \frac{c_{21}(t)}{c_{11}(t)}\,;\ \alpha^{+} = \frac{c_{21}(t+\Delta t)}{c_{11}(t+\Delta t)}\,;\ \beta = \frac{c_{12}(t)}{c_{22}(t)};\ \beta^{+} = \frac{c_{12}(t+\Delta t)}{c_{22}(t+\Delta t)}$$

$$R_1 = \frac{\alpha}{\alpha^{+}};\ R_2 = \frac{\beta}{\beta^{+}}$$

Assuming that the total number of links of the network remains constant in time, if we have two normal communities, it is expected that the number of links in both of the groups will increase and between them will decrease, so both R_1 and R_2 will be higher than 1. If we have a strategic group, the links inside this group (assumed here to be group 1) will increase, and the links from group 1 to group 2 will decrease at the same rate. Also, the links from group 2 to group 1 and inside the group 2 will tend to remain the same, because the group 2 will continue to see the rest of the network as random nodes, so we expect that R_1 to be higher than 1 and R_2 to be close to 1. So we generated a random graph without specific groups through the SBM algorithm and used this graph as a base at time t=0. Then, we consider two situations: one that this base graph evolved to a network with two groups (strategic and non-strategic) and other where the base graph evolved to a network with two defined communities. So, for each situation we calculated R_1 and R_2, considering that at time t=0 we had a random network without specific groups, and expected to find:

For the network with normal communities: $R_1 > 1$ and $R_2 > 1$;

For the network with the strategic group: $R_1 > 1$ and $R_2 \sim 1$.

As told before, the strategic behavior can occur in social networks. Therefore, as a study case, a social network (citation network) simulated by Sande (2016) was used to verify the increase and/or drop of links within and between the groups during time, analyzing the communities in pairs, to identify the strategic behavior. A computational simulation method was implemented to measure the increase/drop of links to confirm the existence of the strategic group. In this simulated citation network, journals start citing each other to increase their impact factor[2] and consequently increase their importance inside the academic society. So, as an application, this network was used to generate an adjacency matrix to measure the reasons R_1 and R_2 during time, confirming that the strategic behavior occurs.

3 Methodology

First, a Matlab code for SBM by Kevin S. Xu (2015) was used to generate A for graphs with two numbers of nodes: n= 400 and n=2000. The results were similar for both networks. Basically, we followed the next steps described below.

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[2] The impact factor (IF) of an academic journal is a measure reflecting the yearly average number of citations to recent articles published in that journal (GARFIELD, 2006).

- First step:
  To generate partition vector *c*:
  - Inputs: *a* = percentage of the nodes that are strategic (*a*=0:0.1:1);
  - c_{1xn} – the first a% of the nodes belongs to strategic group 1 and the others to non-strategic group 2.

- Second step:
  To generate *A*:
  - Define matrix *P* of probabilities;
  - Enter with: *c*, P, indicating that the network is directed (true).
- Third step:

To vary the configuration of the network changing the position of the elements in the partition vector *c*. Each new vector is calculated with a similarity degree *x* in relation to the first vector *c* proposed (*x*=0:0.1:1). For each new vectors *c*, the values of Q_1 and Q_2 were calculated.

As a final output the graphics of Q_1 and Q_2 for each value of *a* were obtained (varying the number of strategic nodes). The values of Q_1 and Q_2 were plotted in the same graphic and 11 graphics were obtained (*a*=0:0.1:1).

As a result, to identify the strategic behavior, the maximum value of Q_1 and the minimum value of Q_2 should be in the network configuration obtained with the highest similarity (*x*=1).

After that, we used also a Matlab code to calculate the reasons R_1 and R_2 based on a random network at time *t*=0. First a random matrix was generated through the SBM model with a symmetric matrix P, then the networks with a strategic group and with two communities was generated also through the SBM model with matrices P_{str} and P_{com}, respectively, and finally R_1 and R_2 were calculated for both of the networks.

$$P = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix};\ P_{str} \begin{bmatrix} 0.12 & 0.1 \\ 0.087 & 0.1 \end{bmatrix};\ P_{com} \begin{bmatrix} 0.12 & 0.087 \\ 0.087 & 0.1088 \end{bmatrix}$$

As a last stage, the calculation of the R_1 and R_2 during time were applied to Sande`s simulated citation network with 400 nodes (journals), where 20% of them were considered to follow the strategic behavior. This network was simulated in a period of 30 simulations cycles, where in the first 11 cycles (considered the base period, *t*=0 until *t*=10) we have a random network with no specific group, in the next 10 cycles (*t*=11 until *t*=20) we have 20% of the nodes following the strategic behavior and in the last 10 cycles (*t*=21 until *t*=30) we have again a random network with all of the nodes without any specific behavior (Figure 1).

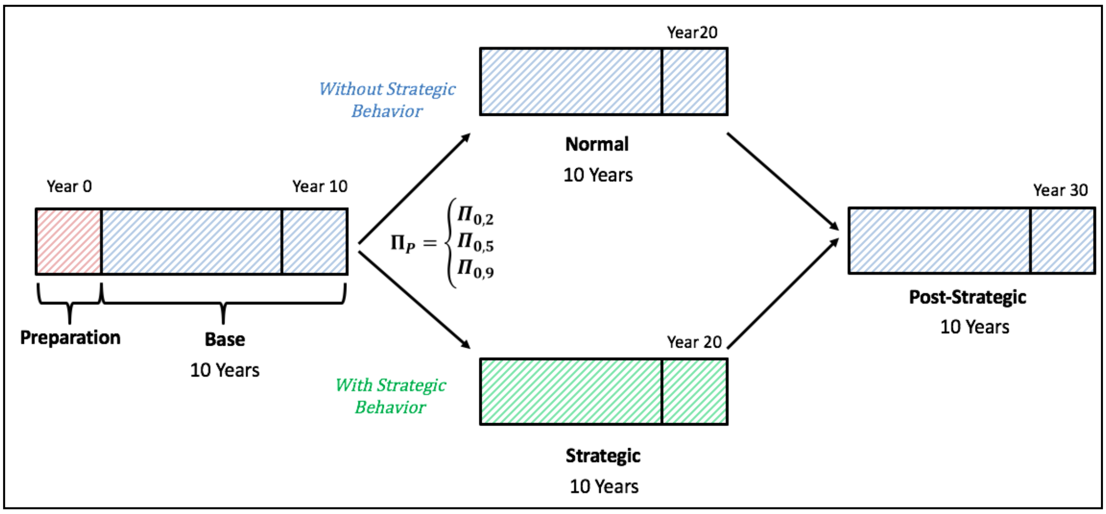

Figure 1 – Simulated network with 20% of the nodes with strategic behavior
Source: BARROS, 2016

4 Results

Some matrices of adjacencies *A* are showed in Figure 2, showing the network configuration for some values of *a*, being able to visualize the graph divided in two groups.

As expected, we found that the values of Q_1 and Q_2 increases and decreases with *x*, respectively (Figure 3), also they were found to be symmetric ($Q_1 = -Q_2$), confirming that the modularity concept does not identifies the strategic behavior.

This confirms that the detection of groups in a directed network depends only of the one value of "submodularity", showing the modularity concept is able to detect sub graphs, but cannot specify if this group is a normal community or a strategic group.

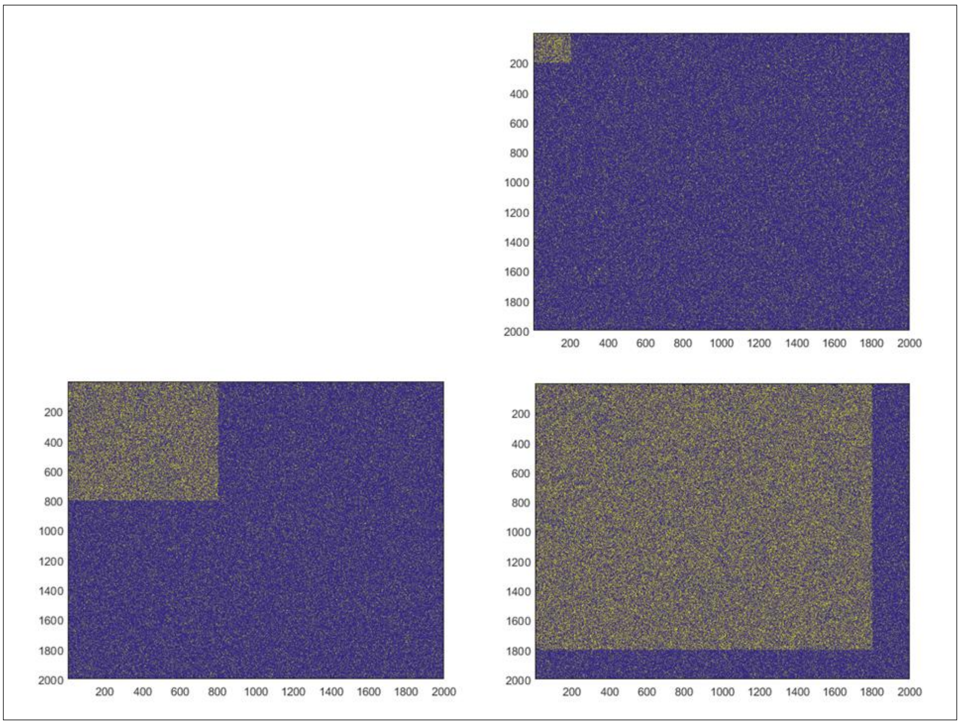

Figure 2 – Matrix A generated with the SBM for values of a=0.1, 0.4 and 0.9

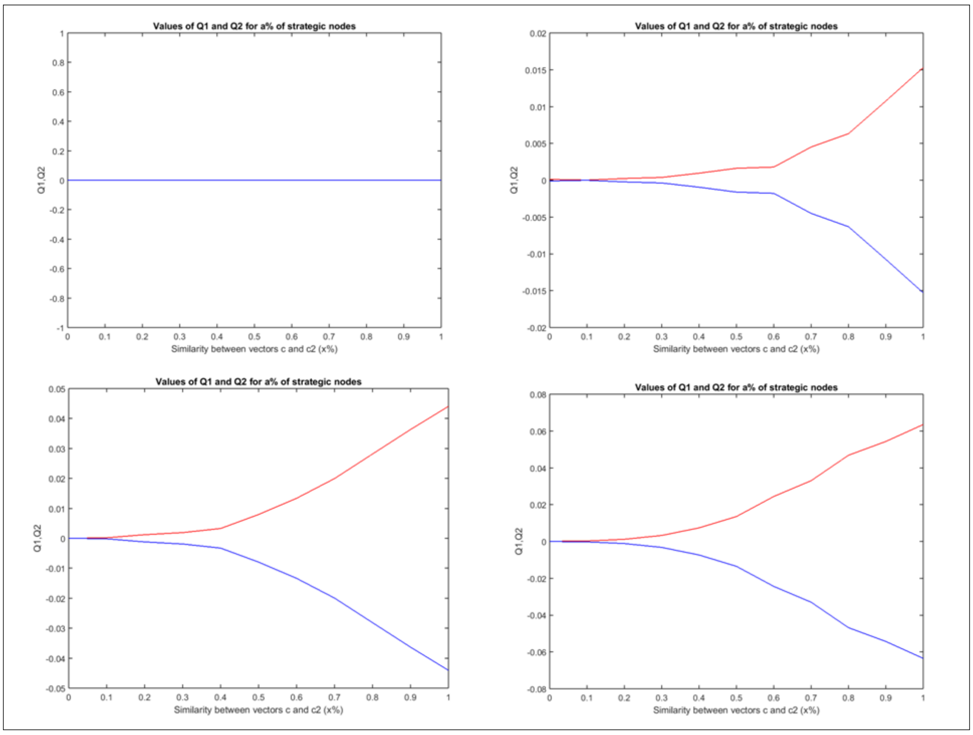

Figure 3 – Graphics of the values of Q1 and Q2 varying with x for some values of a (a=0.0, 0.1, 0.2 e 0.3)

As for the reasons R_1 and R_2 it was found that:

Reason	Community	Strategic
R_1	1.3778	1.3799
R_2	1.2473	0.99136

Table 1 – Values of R1 and R2 for the networks with communities and with a strategic group

As we can see, results were as expected: for the network with communities: $R_1 > 1$ and $R_2 > 1$; and for the network with the strategic group: $R_1 > 1$ and $R_2 \sim 1$.

Finally, for the simulated citation network where the method was applied we can see from Figures 4 and 5 that in the period where the nodes assume the strategic behavior (t=11 until t=20), the values of R_1 and R_2 are as expected ($R_1 > 1$ and $R_2 \sim 1$). Also for the nodes that do not assume this behavior at any time, the reasons are as expected too ($R_1 \sim 1$ and $R_2 \sim 1$).

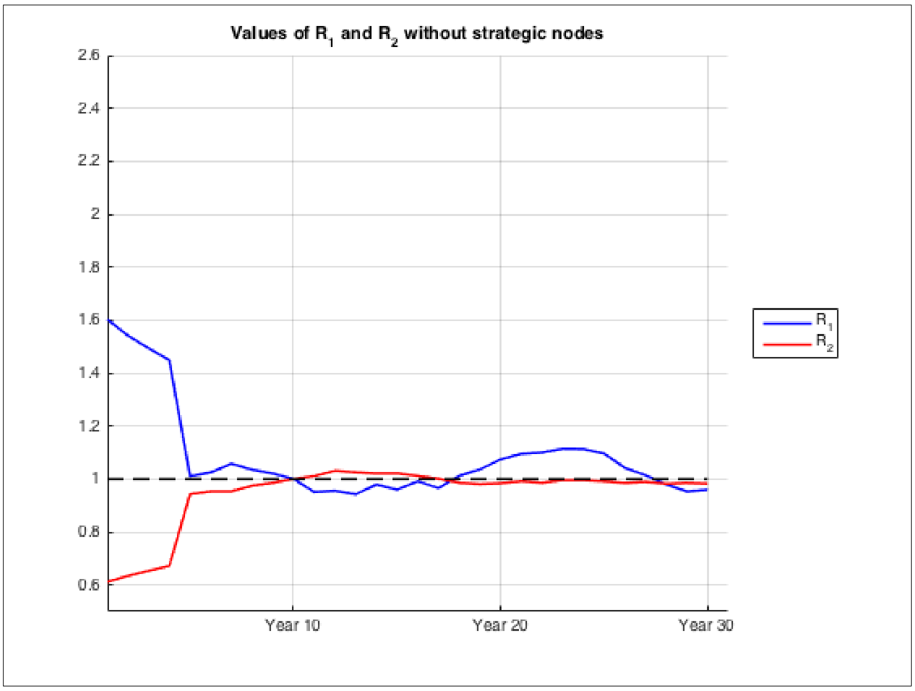

Figure 4 – Values of R_1 and R_2 for nodes not following the strategic behavior

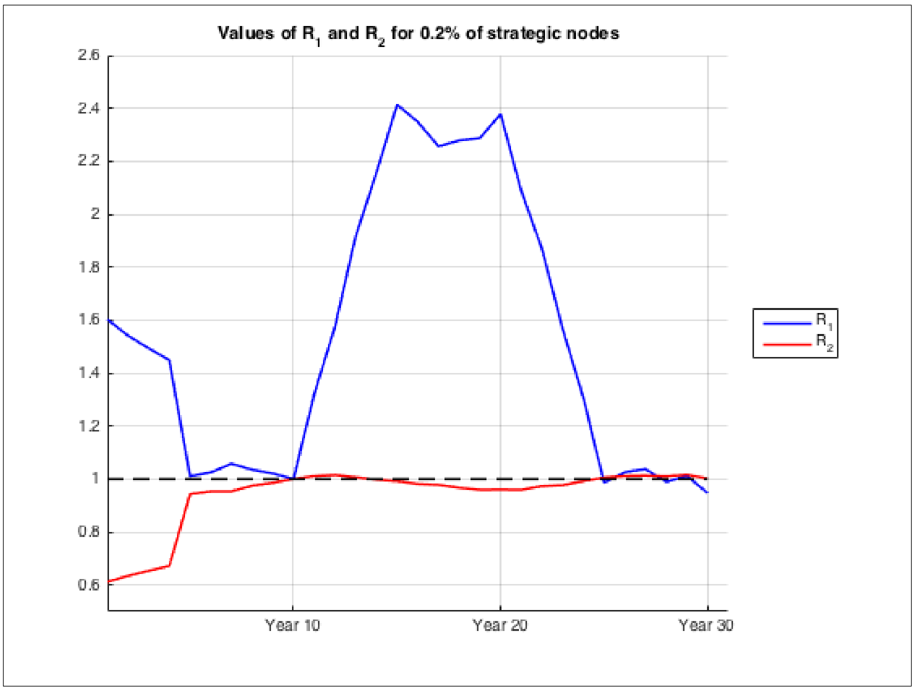

Figure 5 – Values of R_1 and R_2 for the 20% of the nodes following the strategic behavior

5 Conclusions and future studies

The results showed that the identification of the two groups depends only on one value of “submodularity” ($Q_1 = –Q_2$). Also the adapted concept of modularity suggested here demonstrates to be effective to identify two groups in a network, although it does not specify if they adopt the strategic behavior suggested here or not.

Even though the strategic group was not differentiated from a community by the modularity neither by the "submodularity" concept proposed here, nothing can be concluded about the strategic behavior, since it is plausible to happen in social networks, where a strategic group can be considered as a secret society, e.g. the Masonry and sororities in universities and colleges.

Since this kind of behavior is not detected by the usual method of community identification, this study introduces a new method and implemented an algorithm for identifying the strategic behavior in networks: the observation of the graph in different time periods to calculate the reasons R_1 and R_2 based on the increase/drop of links inside and between the groups in a network. It showed to be effective in identifying communities and also the strategic behavior, if this behavior is previously known.

As a practical application, it was used a simulated citation network with certain premises such as strategic behavior, where the method proposed here clearly identified this specific behavior when observing the network during time.

Regarding to future studies, it is suggested to apply this method in some real social networks to find if the strategic behavior occurs and if the method is efficient in identifying it (without previously knowing it).

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